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Many-Quark Interactions: Large-N Scaling and Contribution to Baryon Masses

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Abstract: Starting from an effective Hamiltonian modeling, of a baryon made of N identical quarks in the large-N approach of QCD, we obtain analytical formulas, allowing to estimate the contributions of multiquark interactions to the baryon mass. The cases of vanishing (mass spectrum) and non-vanishing (baryon melting) temperatures are treated.

Keywords: baryons; large-N QCD; multiquark interactions; mass formula; potential model

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1. Introduction

If quantum chromodynamics (QCD) is, indisputably, the theory of strong interaction, many different approaches must be explored, to study this theory in the domain of hadron spectroscopy. Among the most efficient and popular ones are the constituent models and the large-N picture. In the first models, constituent quarks appear, such as "dressed" current quarks, and interact via potentials, simulating the exchanges of virtual gluons and quark–antiquark pairs [1]. The recent detection of baryons made of one or two heavy (c or b) quarks has led to a revived interest in constituent models, such as, e.g., the hypercentral Constituent Quark Model [2,3]. Note that such models are relevant in the field of light baryon spectroscopy as well [4]. In the large-N picture, the SU(3) group of QCD is replaced by SU(N), allowing an expansion for the properties of the theory in powers of $1/N$, which is treated as a small parameter [5]. It appears quite natural to try to combine these two different methods ,to gain new insights about the hadron physics.

It has been shown in [6–11] that constituent models were able to match the arbitrary coefficients, appearing in the large-N baryon mass formulas, which shows the complementarity of both frameworks. However, the analysis of the latter works focuses on one- and two-body interactions. In this work, such a methodology is used to study multiquark interactions, within the baryon. Simple models, with baryons made only of identical quarks, are used to highlight the main effects.

Various large-N versions of QCD exist, in which quarks can be in different colour representations [12,13]; we, here, consider quarks in the fundamental representation, *i.e.* 't Hooft limit [14]. It is known, from the $\mathrm{SU}(2N_f)$ symmetry of baryons at large-N, that the mass formula is of the form [15]

$$M = N \sum_{h=0}^{N} \frac{c_h}{N^h} \mathcal{O}_h, \tag{1}$$

with $\mathcal{O}_h$. an h-quark spin-flavour operator, whose eigenvalue is of order 1, and, also, c_h a number of order 1. Spin-dependent h-quark interactions are suppressed, at least in $1/N^h$, and will not be investigated in the present work, where we focus on spin-independent h-quark interactions, i.e., interactions bringing a contribution to c_0. In a spin-flavour independent approach, the ground-state mass of a baryon made of N quarks of the same flavour can, a priori, be expressed as

$$M = NK + \sum_{h=1}^{h_{\max}} \binom{N}{h} E_h, \tag{2}$$

where K is the average kinetic energy of one quark (including its rest energy), and where E_h is the average potential energy of an interaction involving h quarks (with $h_{\max}$ finite), this energy is being weighted by the binomial coefficient $\binom{N}{h}$.

It has been said that three (or more)-quark interactions "simply renormalise the average Hartree potential" in a mean field approximation for two-body interactions only [5]: the necessary technical complications for their inclusion in a quark model can, thus, be avoided, in principle, since they can be absorbed in a redefinition of the parameters. Nevertheless, it has been shown in [16], for $N = 3$, that three-quark interactions, going beyond one-gluon-exchange (OGE), have relevant physical features, such as increasing the mass gap between colour-singlet baryonic states and unphysical coloured ones, or lifting mass degeneracies in tetraquarks. Only a few works have addressed the problem of three-quarks interactions in quark models [16–18], while, to our knowledge, the question of multiquark interactions in large-N QCD has only been studied in [19]. In this last work, it is concluded that h-quark interactions ($h > 2$) are suppressed by a relative factor $h!$ and can, then, be neglected. This result is obtained with a wave function computed in the Hartree approximation, as in [5]. We want to reconsider this problem by using the envelope theory (ET), which is a method to compute approximate but reliable solutions for many-body quantum systems [20–22]. It is quite easy to implement, if all the particles are identical, and the computation cost is independent of the number of particles. Moreover, analytical upper or lower bounds can be obtained in favourable situations [23]. This will be the case for the systems considered here.

The Hamiltonian, for baryons in the large-N approach, is given in Section 2. The dependence on N for the parameters are fixed in Section 3. One-gluon exchange and three-quark interactions are, respectively, treated in Section 4 and Section 5. While in these sections, baryons are, always, considered at vanishing temperature, their behaviour above the deconfinement temperature T_c is examined in Section 6. Concluding remarks are given in Section 7.

2. Baryon Mass Formula

We aim at deriving the mass formula (2), which is based on intuitive mean-field arguments, from an effective theory based on constituent quarks. A generic quark model of baryon, made of N identical quarks at vanishing temperature, including one- and h-quark interactions, may be defined by the Hamiltonian (as usual, u and d quarks can be considered as two projections of the same quark, through isospin)

$$H = \sum_{i=1}^{N} \left[K(|\vec{p}_i|) + U(|\vec{x}_i - \vec{R}|) \right] + \sum_{\{i_1,\ldots,i_h\}}^{N} V_h(r_{\{i_1,\ldots,i_h\}}) \tag{3}$$

with K the kinetic energy, U the one-body interaction simulating the confinement (with $\vec{R}$ being the centre of mass position) and V_h a h-body potential. $\vec{x}_i$ and $\vec{p}_i$ are the individual conjugate variables. The h-body variables are defined by

$$r_{\{i_1,\ldots,i_h\}} = \sqrt{\sum_{i<j}^{\{i_1,\ldots,i_h\}} r_{ij}^2} \quad \text{and} \quad r_{ij} = |\vec{x}_i - \vec{x}_j|, \tag{4}$$

where the symbol $\{i_1, \ldots, i_h\}$ denotes a set of h quarks among the N quarks in the baryon, with $i_1 < \cdots < i_h$. The sum $\sum_{\{i_1,\ldots,i_h\}}^{N}$ runs over the different sets $\{i_1, \ldots, i_h\}$, while the sum $\sum_{i<j}^{\{i_1,\ldots,i_h\}}$ runs over the different pairs in a particular set $\{i_1,\ldots,i_h\}$. We assume that h is finite, that is, not of order N. This kind of many-body interaction may not be the most general h-body interaction allowed by symmetry arguments, but it has, already, been used in hadronic physics [16,17] and for systems of cold atoms [24]. As is usual in constituent models and in large-N QCD, baryons are treated as stable clusters of quarks, in first approximation [1,5].

As shown in [25], by resorting to the ET, an approximate mass formula may be found, by solving the following set of Equations ($\hbar = c = 1$):

$$\begin{aligned} M &= N\,K\left(\frac{Q}{Nx_0}\right) + N\,U(x_0) + \binom{N}{h} V_h(\psi_h x_0), \\ \frac{Q}{Nx_0}\,K'\left(\frac{Q}{Nx_0}\right) &= x_0\,U'(x_0) + \binom{N}{h}\frac{\psi_h}{N} x_0 V_h'(\psi_h x_0), \end{aligned} \tag{5}$$

where a prime denotes the derivative of the functions $K(z)$, $U(z)$ and $V_h(z)$, with respect to their real argument z. x_0 is a parameter specific to ET and has to be eliminated from the second equation (5), then injected in the mass formula. It is linked to the size of the system [22,23,25]. If the parameter x_0 can be, analytically, extracted from its defining equation, a closed form is obtained for M, as a function of the global quantum number Q.

The global quantum number reads

$$Q = \sum_{j=1}^{N-1}\left(2n_j + \ell_j + \frac{3}{2}\right), \tag{6}$$

and

$$\psi_h = N\sqrt{\frac{h(h-1)}{N(N-1)}}. \tag{7}$$

The formula for M in (5) is of the form (2), as expected, and it is valid for the ground state as well as for the excited states, a particular state being defined by the global quantum number Q. The quantum numbers $\{n_j, \ell_j\}$ are associated with the $N-1$ internal Jacobi coordinates, and the form of Q originates from the exact solutions of translation-invariant N-body harmonic oscillator Hamiltonians, on which the ET relies [26,27]. Quark spin is, obviously, neglected, but, since the singlet-colour wave function is totally antisymmetric, quarks can be seen as "quasi-bosons", with a totally symmetric spatial-spin-flavour part. For instance, the ground state is a totally symmetric wave function, and the spin-flavour part must, also, be totally symmetric. For u and d quarks, all states, with the same value for the spin and the isospin, from $1/2$ to $N/2$, can be built.

For excited states, the situation is trickier, as the orbital wave function is not symmetrical. One way this was, previously, dealt with is by building a mixed-symmetric spatial-wave function, which has to be coupled with a mixed-symmetrical spin-flavour function [28–31]. To build an N-body internal spatial wave function, with a definite symmetry,is a difficult task, even in the case of $N = 3$ [32]. For instance, in the framework of the ET, it is necessary to determine what values of the global quantum number Q are allowed, taking into account a fixed symmetry.

Confinement in baryons has been, extensively, studied within quark models, since the celebrated work [1]. For $N = 3$, the confinement potential is a Y-junction, i.e., a potential of the form $\sigma\sum_{i=1}^{3}|\vec{x}_i - \vec{x}_J|$, where σ is the string tension, and where $\vec{x}_J$ is the position of the string junction, located at the Steiner (or Toricelli) point of the triangle made by the quarks. $\vec{x}_J$ can be identified with the center of mass $\vec{R}$, in good approximation [33]. We, therefore, perform this replacement at arbitrary N, also, and use a potential of the form $\sigma\sum_{i=1}^{N}|\vec{x}_i - \vec{R}|$, as in [5]. In other words, we use the one-body form of the confinement in

Hamiltonian (3), with $U(x) = \sigma\, x$. Explicit ($h \geq 2$)-quark interactions will be discussed, later. Our ansatz for V_h is a scaling of the form

$$V_h = \mathcal{C}_h\, g_h v_h, \tag{8}$$

with g_h, a coupling constant a priori depending on the strong coupling constant g (see next section), $\mathcal{C}_h$, a colour factor involving SU(N) Casimir operators and v_h, the space part of the average potential energy.

3. Large-N Scaling

The large-N behaviour of our model can, now, be investigated. In the following, the symbol $\sim$ is used for a quantity evaluated for $N \to \infty$ (possibly up to a multiplicative constant). For states with finite quantum numbers $\{n_j, \ell_j\}$, $Q/N \sim 3/2$ ($Q = 3(N-1)/2$ for the fully symmetrical ground state). Moreover, $\psi_h \sim \sqrt{h(h-1)}$. As h is finite, it is useful to recall that $\left(\dfrac{N}{h}\right)^h \leq \dbinom{N}{h} \leq \dfrac{N^h}{h!}$. Hence, $\dbinom{N}{h} \sim a_h\, N^h$ where a_h does not depend on N.

Assuming the Casimir scaling [34–37], which follows from a strong coupling expansion in lattice QCD, one has $\sigma = \frac{N^2-1}{2N} g^2 \Omega$ with Ω a constant [38,39]. Since $g^2 = \lambda/N$ with λ the 't Hooft coupling, σ is constant at large-N.

The power in N of the coupling constant g_h may be found, as follows. Any Feynman diagram involving $h > 2$ quarks must contain h quark–antiquark–gluon vertices, so that each quark emits a gluon line: a g^h factor is present. Then, the quarks have to be "linked". Linking two quarks, by the exchange of one gluon, costs at least one quark–antiquark–gluon vertex or a three-gluon vertex with one outgoing gluon line, i.e., a factor g. There are at least $h-2$ such links, so the dominant Feynman diagrams are proportional to g^{2h-2}, in agreement with [5,40]. One can, therefore, state that the h-quark coupling constant is such that

$$g_h \sim g^{2h-2} \equiv \bar{g}_h\, N^{1-h} \tag{9}$$

for the leading-order diagrams. By definition, $\bar{g}_h$ does not depend on N. Notice that two consecutive three-gluon vertices may be replaced by a single four-gluon vertex, which does not change the order in g. Equation (9) is, thus, valid for any leading-order diagram.

Gathering all these results, when $N \to \infty$, Equations (5) become

$$\begin{aligned} \frac{M}{N} &\sim K\left(\frac{3}{2x_0}\right) + \sigma x_0 + a_h\, \mathcal{C}_h\, \bar{g}_h v_h\left(\sqrt{h(h-1)}x_0\right), \\ \frac{3}{2x_0}\, K'\left(\frac{3}{2x_0}\right) &\sim \sigma x_0 + a_h \sqrt{h(h-1)}\, x_0 \mathcal{C}_h\, \bar{g}_h v_h'\left(\sqrt{h(h-1)}x_0\right). \end{aligned} \tag{10}$$

It follows form Equations (10), that $\mathcal{C}_h$ has to be constant at large-N, at leading order, for the baryon mass to be of order N. The criterion $\mathcal{C}_h \sim 1$, being a sufficient condition to recover the expected baryon mass scaling at large N, may actually be seen as a selection criterion in model building: h-quark interactions leading to a different scaling have to be ruled out.

We will, now, check that known explicit interactions are consistent with that requirement. Recall that $\frac{1}{h^h} \leq a_h \leq \frac{1}{h!}$. The fact that h-quark one-gluon exchange interactions bring a contribution of order $N/h!$ to the baryon mass was, already, suggested in [19], for heavy baryons. Here, we extend this result to arbitrary baryons made of identical quarks and show that the actual suppression is even stronger than $1/h!$.

4. Two-Body Interactions

At leading order in N, two-quark interactions are OGE processes, typically associated with a Coulomb potential of the form

$$\mathcal{C}_{2\,\mathrm{OGE}} \sum_{i<j}^{N} \frac{\alpha_s}{|\vec{x}_i - \vec{x}_j|} \quad \text{with} \quad \mathcal{C}_{2\,\mathrm{OGE}} = \frac{1}{2}\left(C_2^2 - 2C_2^1\right), \tag{11}$$

where $\alpha_s = \frac{g^2}{4\pi}$ and where C_p^k is the eigenvalue of the SU(N) Casimir operator of order p, for the totally (anti)symmetric representation, with k indices. Explicit formulas for C_p^k are given in Appendix A.

For colour-singlet baryons, two quarks are always in the antisymmetrical representation and, according to Equation (A2),

$$\mathcal{C}_{2\,\mathrm{OGE}} = -\frac{N+1}{2N}. \tag{12}$$

This interaction is of order 1 at large-N, as required. Its iteration between h quarks, as sketched in Figure 1, does, obviously, not change the order of the contribution.

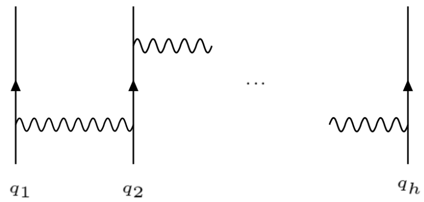

Figure 1. One-gluon exchange process involving h quarks.

One- and two-quark interactions bring a contribution of order N to the baryon mass, as it has been checked within a Hamiltonian approach, with relativistic quark kinetic energy and linear plus Coulomb potential in [7]. It is worth mentioning that one-gluon exchange processes, also, lead to hyperfine corrections (spin-orbit, spin-spin, etc.). It has been shown, in [8], that spin-spin corrections, only, bring a contribution of order $1/N$ to the baryon mass, in agreement with (1), and can, then, be neglected in a first approximation.

5. Three- and More-Quark Interactions

We now focus on three-quark interactions, beyond OGE; an example is shown in Figure 2. The natural choice made in [16,17], for such interactions between quarks q_1, q_2 and q_3, involves the colour operator $d_{abc} T^a_{q_1} T^b_{q_2} T^c_{q_3}$, where d_{abc} are the fully symmetrical coefficients of SU(N). For colour-singlet baryons, three quarks are always in the antisymmetrical representation and, according to Equation (A4),

$$\mathcal{C}_3 = \frac{(N+2)(N+1)}{2N^2}. \tag{13}$$

which is of order 1 at large-N, as required.

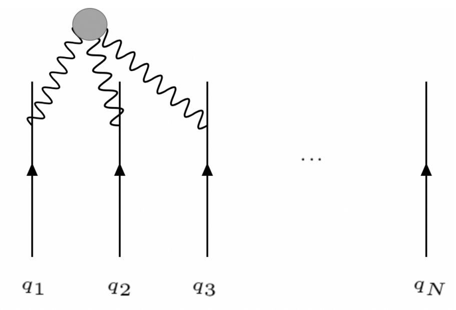

Figure 2. Typical three-quark interaction, beyond OGE.

The dynamics of three-quark interactions can be appraised, through a generalisation of the model proposed in [17,18], which is an attempt to study a nonperturbative three-body interaction, of the confining type. We, first, chose a kinetic energy of the form

$$K(p) = \mu_\alpha\, p^\alpha + m, \tag{14}$$

that can accommodate for both massless ($\alpha = 1$, $\mu_1 = 1$) and massive ($\alpha = 2$, $\mu_2 = \frac{1}{2m}$) quarks. We can expect that the most dramatic effect of three-quark interaction will manifest itself in confinement. So, we set $V_3(z) = \mathcal{C}_3 g_3\, z$, in agreement with the choice $U(z) = \sigma\, z$. In this case, one finds from (5) that an upper bound for the mass per quark is given by

$$\frac{M}{N} = m + (1+\alpha)\left[\frac{\mu_\alpha^{1/\alpha}}{\alpha}\sigma_3\frac{Q}{N}\right]^{\alpha/(1+\alpha)}, \tag{15}$$

with

$$\sigma_3 = \sigma + \binom{N}{3}\frac{\mathcal{C}_3}{N} g_3 \psi_3. \tag{16}$$

Note that upper bounds for three massless quarks interacting via linear and Coulomb potentials are computed, in the framework of the ET, with relative errors around 15% [23]. It appears, clearly, that the three-quark interaction can be absorbed in a shift of σ, with this shift being of order 1 at large N: $\sigma_3 \sim \sigma + \frac{\bar{g}_3}{2\sqrt{6}}$. The mass formula (15), although being approximate, predicts baryon Regge trajectories for light quarks, i.e., $m = 0$ and $\alpha = 1$.

If $h > 3$, a great number of different structures are possible for the colour operator $\mathcal{C}_h$. All can be built from various combinations of the algebra structure constants f_{abc} and the symmetric coefficients d_{abc}. The equivalent of Equation (15) with the ansatz $V_h(z) = \mathcal{C}_h g_h z$, mimicking a h-body confinement, is, then, obtained by replacing σ_3 by σ_h with

$$\sigma_h = \sigma + \binom{N}{h}\frac{\mathcal{C}_h}{N} g_h \psi_h. \tag{17}$$

If we assume that for all values of h, $\mathcal{C}_h \sim 1$, as is the case for $h = 2$ and 3, the string tension σ is, simply, modified by a quantity $\sim a_h\sqrt{h(h-1)}\,\bar{g}_h$. With the peculiar choice $V_h(z) \sim z$, the effect of multiquark interactions can, always, be absorbed in a redefinition of σ.

6. Baryon Melting at Finite Temperature

Up to now, we focused on baryons at $T = 0$. The existence of baryons above the deconfinement temperature T_c has been suggested in [41], as well as the relevance of many-body force in nucleon clustering, near the QCD critical point [42]. Starting from a Hamiltonian, with nonrelativistic kinetic energy and a finite-range two-body potential, fitted on SU(3) lattice QCD, it has been shown that baryons may exist up to 1.6 T_c. May three (or more) quark interactions favour the binding of baryons, above T_c? Although this topic deserves calculations that include, properly, the continuum sector, our framework may provide some indications. Above the deconfinement temperature, it can be assumed that all quarks are massive, since they gain a thermal mass that can be computed using hard-thermal-loop theory, for example, [43]. In this last work, the thermal quark mass m_q, at a temperature T, is found to be $m_q^2 = \frac{N^2-1}{2N^2}\lambda T$, i.e., independent of N, at dominant order.

In a quark–gluon plasma, many colour channels are possible for a baryon. However, $\mathcal{C}_2 \sim 1$ and $\mathcal{C}_3 \sim 1$, for all possible representations (see Appendix A). Let us consider a nonrelativistic Hamiltonian, with a particular h-quark interaction

$$H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} - |\mathcal{C}_h| N^{1-h} \bar{g}_h \sum_{\{i_1,\ldots,i_h\}}^{N} v_h(r_{\{i_1,\ldots,i_h\}}), \tag{18}$$

where the N-dependence (9) of g_h has been used. We assume that $\bar{g}_h > 0$ and that v_h is a monotonic globally positive function, with $v_h(x \to \infty) = 0$, to mimic finite-range interactions, above T_c. It is not relevant to consider massless quarks with short-range attractive potentials, which cannot bind such particles. Moreover, as mentioned above, it is physically expected that light quarks are dressed with a constituent mass, by the surrounding bath of free quarks, antiquarks and gluons [41,43].

Starting from (18), it can be obtained from [25] that the upper bounds of critical coupling constants—the values of $\bar{g}_h$ below, which a bound state with quantum number Q melts [44]—for such potentials are given by

$$\bar{g}_h = \frac{Q^2}{2m|\mathcal{C}_h| x_0^2 v_h(x_0)} \frac{N^h}{\binom{N}{h}} \frac{h(h-1)}{N(N-1)}, \tag{19}$$

where the quantity $x_0^2 v_h(x_0)$ depends only on the form of v_h and is determined by the relation $x_0 v_h'(x_0) + 2v_h(x_0) = 0$. Note that the accuracy of the ET, for nonrelativistic systems with two-body attractive Gaussian potentials, has been tested as very good, until $N = 20$ [23]. Since $Q^2 \sim N^2$, it is, readily, seen that the criterion $|\mathcal{C}_h| \sim 1$ leads to $\bar{g}_h \sim 1$, hence, to the conclusion that the melting temperature does not depend on N, at dominant order.

The more $\bar{g}_h$ is small, the more the interaction will lead to a high temperature for baryon melting, since $\bar{g}_h$ is, a priori, a decreasing function of T, from renormalization arguments (see, e.g., [45], where the temperature T is taken as the energy scale). Let us compare the critical constants for a given state (Q and m fixed) but for different h and a common function v_h $\forall$ h. Equation (19) leads to

$$\frac{\bar{g}_{h+1}}{\bar{g}_h} = \frac{|\mathcal{C}_h|}{|\mathcal{C}_{h+1}|} \frac{N}{N-h} \frac{(h+1)^2}{h-1}. \tag{20}$$

As an illustration, one can write that, for baryons in a colour singlet,

$$\frac{\bar{g}_3}{\bar{g}_2} = 9 \frac{N^2}{N^2-4}. \tag{21}$$

In other words, since $\bar{g}_3$ is well larger than $\bar{g}_2$, including three-quark interactions, it is not expected to significantly increase the baryons melting temperature.

7. Summary

The dependence on N, of some possible many-quark interactions in baryons, are investigated in the framework of the large-N approach of QCD, with simple constituent models for quarks, in the fundamental representation. In particular, h-quark interactions, depending on the quark positions through (4), are considered. Analytical information is obtained, below and above the deconfinement temperature T_c ,with simple constituent quark models solved with the envelope theory.

Below T_c, baryons are bound states of N quarks, with mass proportional to N, as $N \to \infty$ ($M \sim N$). This sets's constraints on the behaviour with N, of the possible colour h-body operators $\mathcal{C}_h$, can be used in a constituent framework. Computations for $h = 2$ and 3 show that these operators behave as constants, when $N \to \infty$ ($\mathcal{C}_2 \sim 1$ and $\mathcal{C}_3 \sim 1$), which is compatible with $M \sim N$. Moreover, it is shown that $M \sim N$ remains true, if $\mathcal{C}_h \sim 1$ for $h > 3$. These are indications that the relevant structures for $\mathcal{C}_h$ with $h > 3$ must be chosen, in order that $\mathcal{C}_h \sim 1$. When an h-quark-confining interaction is added to the one-body confinement, our results suggest that the latter interactions, mostly, result in a rescaling of the string tension.

Above T_c, our calculations show that the melting temperature of colourless baryons (if any) is independent of N, at leading order, and that the inclusion of multiquark interactions does not stabilize deconfined baryons.

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Appendix A. Casimir Operator for Two and Three Quarks

The eigenvalues C_p^k of the SU(N) Casimir operator of order p, for the totally antisymmetric and totally symmetric representations with k indices, are give,n respectively, by [46]

$$C_p^k = \frac{k(N \pm 1)(N \mp k)}{2N^p(N \mp k \pm 1)} \left[(N \pm 1)^{p-1}(N \mp k)^{p-1} - (-k)^{p-1} \right], \tag{A1}$$

with a normalisation, such that $C_2^1 = 4/3$ for $N = 3$, which is the usual normalisation in constituent models [1,7,8]. Two quarks can, only, be in two colour representations: antisymmetric (A), which is the singlet one, and symmetric (S). The corresponding values for $\mathcal{C}_{2\,\mathrm{OGE}}$ are given by

$$\begin{aligned} \mathcal{C}^A_{2\,\mathrm{OGE}} &= -\frac{N+1}{2N}, \\ \mathcal{C}^S_{2\,\mathrm{OGE}} &= \frac{N-1}{2N}. \end{aligned} \tag{A2}$$

Both colour coefficients are of order 1, when $N \to \infty$.

The procedure presented in [17] for the computation of the Casimir operator for three quarks in SU(3) can be generalised to SU(N), and one has

$$\mathcal{C}_{3q} = d_{abc} T^a_{q_1} T^b_{q_2} T^c_{q_3} = \frac{1}{6} \left[C_3^{3q} - \frac{3}{2} \frac{N^2 - 4}{N} C_2^{3q} + 6 C_3^{1q} \right], \tag{A3}$$

where $C_3^{3q} = d_{abc} \sum_{q_1,q_2,q_3=1}^3 T^a_{q_1} T^b_{q_2} T^c_{q_3} = (C_3^3 - C_2^3)/2$, $C_3^{1q} = d_{abc} T_1^a T_1^b T_1^c = (2C_3^1 - C_2^1)/4$ and $C_2^{3q} = \delta_{ab} \sum_{q_1,q_2=1}^3 T^a_{q_1} T^b_{q_2} = C_2^3/2$ as well as where the values C_p^k are given in (A1). Three quarks can only be in three colour representations: antisymmetric (A), which is the singlet one, symmetric (S) and mixed-symmetric (M). The corresponding values are given by

$$\begin{aligned}\mathcal{C}_{3q}^{A} &= \frac{(N+2)(N+1)}{2N^2},\\ \mathcal{C}_{3q}^{S} &= \frac{(N-2)(N-1)}{2N^2},\\ \mathcal{C}_{3q}^{M} &= -\frac{N^2-4}{4N^2}.\end{aligned} \tag{A4}$$

For $N = 3$, these values are in agreement with the ones obtained in [16–18]. All these colour coefficients are of order 1, when $N \to \infty$.

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