HIGH LEVEL CONSTRAINTS WEIGHTING ON THE POSSIBLE SHAPES KNOWLEDGE CAN TAKE ON

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ABSTRACT: In this paper, we present a study that can be seen as a partial bridge between didactic researches centered around epistemological concerns like epistemological obstacles and sociological approaches that can be used in educational sciences. To do so we use an anthropological approach to the didactic of mathematics, the anthropological theory of the didactic (ATD) and in particular its "scale of levels of didactic codetermination". This enables us to bring into the fore a set of constraints emanating from institutions located at a high level of this scale. The two institutions we are considering are the modern mathematics reform and the counter-reform that followed. The constraints generated by these institutions form a dense network that compel the possible shapes mathematical knowledge can take on. Such a stringency leads to some hybridization of the deductive and argumentative architecture of courses subdued to these constraints. The concepts become two sided. They bear at the same time a mathematical side, the most visible one, that has a mathematical legitimacy and another less visible one whose role is to allow these concepts' ecological viability. The implicit and unconscious shifts between these two polarities tend to reinforce cultural mimetic encounters of knowledge based on ostensive practices.

Key words: ATD, epistemology, anthropology, mathematics, economy, constraints

INTRODUCTION

This paper presents an ongoing research. It is short since we formally began to work on this topic only 2 months ago. Moreover, we are not yet able to fully position ourselves among the existing literature because of the specific context in which this research emerged. It first began as a teacher questioning and not a researcher interrogation: "How is it that in spite of all our efforts, the failure rates to our course is so high?". From this initial teacher perspective, an answer had to be found, not in scientific papers, but by merely "looking" at our high school's ordinary working.

Nevertheless, the study of this question led us to gradually depart from our initial approach in two ways.

- We had to consider a new question, related to the initial one, about high failure rates: the question of understanding antinomies found within our own course. This new question became the central one, the other being subordinated to it.

- We were "forced" to embrace theoretical tools that would allow us to tackle this new question, namely the *Anthropological Theory of the Didactic* (ATD) developed by Yves Chevallard (1992). Most notably we will make use of one of its fundamental tools, the *scale of levels of didactic codetermination*. ATD had indeed the potentiality to endow us with concepts allowing us to formulate hypotheses about the possible shape a mathematical knowledge can take on.

These two elements mark our transition from teachers to researchers.

This paper is structured as follow. We start by giving an account of the salient steps that took us from our initial question of high failure rates, to a scientific approach and the question of antinomies. We then give an example of an antinomy to illustrate what we mean by it and at what level it is located. The section that follows introduces ATD and the tools we need to interpret these antinomies, namely that they are the resultant of two opposing types of institutional constraints. We conclude with the importance of taking into account those types of constraints, if one wants to be able to understand what shape a knowledge can or cannot take on in a mathematical course.

FROM A TEACHER QUESTION TO A RESEARCHER QUESTION

Let us start by first explaining the institutional framework in which our initial question took place. We and three other teachers are giving a mathematical course to 500 first year students in a Belgian high-school[1] in economy, business and management. The course content is very similar to a secondary school curriculum. There is a chapter devoted to the "theory" of functions which limits itself to real function of a single real variable, assorted with a few concepts, those found in secondary school, like, for instance, domain of a function, injectivity, monotonicity. Then come chapters devoted to specific classes of functions (first degree functions, second degree functions, etc.). Among these chapters, one is of special interest for our purpose. We will draw on it, to give an example of what we call antinomies within the course. This chapter deals with exponential, logarithmic and reciprocal functions. Another chapter deals with non-constrained optimization, of one real variable functions, using derivatives (study of the first and second derivatives). Those reminders are broadened out with classical economic applications: budget line, elasticity, equilibrium between supply and demand.

Besides these secondary level chapters, they are two others dealing with topics new too students. One chapter is introducing *elements* of first order logic, mostly logical connectors ($\wedge$, $\vee$, $\neg$, $\Rightarrow$) and quantifiers ($\exists$, $\forall$). Another is introducing linear programming restricted to two variables. These linear programming problems are not solved by the simplex method, but rather using a geometrical approach that consists in calculating an "intersection" between a polygon and a parallel sheaf of lines.

One reason for the limited course content is the great variety of students' profiles. Some didn't do any mathematics for one year or more, while others had 8 hours of mathematics per week. Among these varied profiles, none seems to be more prevalent than others.

The only salient fact we had to face are the high failure rates, between 50% and 75% for the last five years, despite numerous attempts to improve the situation: new exercises with detailed solutions, improved explanation based on recurring errors, decreased number of students in groups, methodology courses part of the curriculum, etc. Our purpose is not to detail all the innovations devised by teachers but rather to highlight the amount of energy invested and the almost zero return, to the point some of our colleagues didn't want to be in charge of the course anymore.

Different leads were envisioned to explain the situation. Might the problem be related to the new knowledge (logic and linear programming)? This hypothesis turned out to be false. Students had as much trouble with secondary school knowledge than with say linear programming. Might the high failures rates be correlated with the number of hours per week students had in secondary school? The answer was negative as well. More hours didn't mean higher success rates. Given those negative answers, a trend among our colleagues emerged: the students themselves had to be the reason for the high failure rates: they are not working enough and they show no real interest in mathematics.

This standpoint cannot be entirely rejected, for it is true that our students are rather immature. For instance, when given (very easy) exercises to prepare, most don't do anything, except wait for the teacher to give the "right" answer. In classroom, they spend more time playing with electronic devices than working on their mathematics. The list of immature behavior is almost endless but it is not our point to be harsh on the students and blame them for everything. Even if we share the idea that they should be more involved in their studies, we were led to take into consideration another dimension that remained hidden for quite some time.

We found antinomies within our own course. We thus had to consider the possibility that these were, at least partly, accountable for students' high failure rates. We moreover thought this lead was all the more credible for several reasons. They were many of them, not just a few. They were scattered throughout the course and they were all related to the course's deductive architecture. That could not be a coincidence.

ANTINOMIES

Let us give an example of an antinomy found in the chapter introducing exponential functions and logarithmic functions as reciprocal of exponentials. We first start by pointing out "strange things" within the chapter then we explain why these "strange things" together form an antinomy.

A first problem is found, right from the start, within the very definitions of these types of functions. The exponential functions are defined as functions of the form:

$$x \mapsto f(x) = C \cdot a^x$$

[1] In the French speaking part of Belgium, a high-school, at least in our setting, designate an institution similar to a university. The specific differences between a high-school and a university is of no interest for our purpose and belongs to Belgian administrative organization.

where $C \neq 0$, $a > 0$ and $a \neq 1$. The problem is that this definition does not satisfy the property:

$$f(x + y) = f(x) \cdot f(y)$$

later announced as a fundamental property of exponentials. Put that way, the contradiction seems blatant. How did it get unnoticed? The way it is dressed in the course is rather tricky and not that easy to detect. The initial definition given to students is that of functions having an exponential growth ($x \mapsto f(x) = C \cdot a^x$). At that point in the course it is not clear if these functions are meant to be exactly the exponential functions or not. It is only by looking carefully at various semantic shifts that we finally have to conclude that functions having an exponential growth are the same as exponentials functions. Without that careful analysis, the problematic connection would have gone undetected.

We now turn to the second definition, that of logarithmic functions. Contrary to the chapter's purpose, they are not explicitly defined as reciprocal of exponentials, but instead, in a rather loose way, using notations that allows students not to integrate this relationship and its relevance:

$$\exp_a(x) = y \iff x = \log_a(y)$$

The reciprocal relationship between exponentials and logarithmic functions is merely stated afterwards, as a property among many others. This leads us to another oddity.

All exponentials and logarithmic functions' properties are on an equal footing. The fact that some are consequences of others is completely silenced. This leveling resonates with the given "proofs" of those properties.

They are no actual proofs of exponential functions' properties. In place, we find formal plays between the notations a^x and $\exp_a(x)$, used to "reformulate" the various properties. The status of these plays is never clarified to the students to the point that some do consider them as genuine proofs.

Getting back again to logarithmic functions, we find this time actual proofs but no unified proof principle is put forward. Each proof is a special case requiring some form of "illumination" on the part of the creator. The reciprocal operator on functions isn't presented as a mean to automatically transport properties from one class of functions to another. The reciprocal operator thus plays no role in the deductive architecture of logarithmic functions, not even in its definition. What is then its purpose in the course?

The above description of the chapter about exponentials and logarithmic functions exemplifies what we label as an antinomy. It is the global way exponentials and logarithmic functions are treated that is antinomic. Looking from a distance, we might be under the illusion that a perfectly ordinary deductive presentation of the subject is given to students. But, exerting a sharper look, reveals "strange details" that, taken as a whole, highlight an evanescent deductive architecture. This is where the antinomy is located: the presence of a deductive architecture looking at the big picture and no deductive architecture looking closer.

This example, characteristic of antinomies found in the course, rises questions. Here are a few, given in no particular order, to illustrate the dynamic antinomies are able to generate.

- (Q1) How is it possible for such an elementary course to be filled with antinomies?
- (Q2) Why are our colleagues left rather unmoved or at best annoyed when we present them with these antinomies?
- (Q3) Why did it take us so much time to detect these antinomies and be able to put them into words?
- (Q4) Why maintain the illusion of a deductive architecture when there is almost none? To what end?

Answers to these questions, formulated as hypothesis, can be given when we introduce the theoretical tools of the next section.

THEORITICAL FRAMEWORK

What kind of theory do we need to address questions 1 to 4? Because of their specific nature, the studied antinomies are not that easy to detect or talk about. They are not, so to speak, "direct" mathematical errors, like an erroneous proof, or a computational mistake. They nevertheless belong to the mathematical sphere. They are related to the relationships teachers have towards mathematics, relationships that end up in a double play around the course's deductive architecture. Based on this characteristic, we felt we had to turn ourselves towards a didactical theory that could take into account the relationships people can have with knowledge. This is precisely some of the things ATD[2] is capable of. A basic tenet of ATD is to consider knowledge as emerging from human practices (Chevallard, 1992). These practices happen in institutions. In ATD's sense an institution can be an institution in an ordinary sense but it is not limited to that. It can be, for instance, a particular school of thought, like bourbakism we shall talk about in the next section. The strength of this approach is to take a step back from epistemologies, like Platonism[3], that consider knowledge, especially mathematical one, as something immutable, independent of any cultural background, at least when its "true essence" has been put to light. Taking that step back allows oneself to formulate hypothesis about antinomies or any didactical problem in terms of institutional disparities. This is what ATD calls *analysis of transposition* (Chevallard, 1991). Such an analysis has been a main tool of action in the French school of didactics, for several decades. This is the theoretical framework which we adopt in this paper.

The analysis of transposition is somewhat formalized relying on the *scale of levels of didactic codetermination* tool (Chevallard, 2005). The basic idea leading to the scale is to take into account the impact on knowledge of institutions located at different strata of society, ranging from a specific mathematical subject, like the definition of exponential functions, to civilization issues. Taking into account all these strata can be seen as a tangible mark of ATD's willingness to be a systemic approach to didactic. An institution located at the level of civilization is for instance the peculiar way Greek mathematicians envisioned numbers. To them, the only genuine ones were the natural numbers. Without taking this relationship to numbers into account, it is hardly possible to understand Greek Mathematics.

The strata are arranged into a scale. The levels of that scale, from the lowest to the highest, are: subject, theme, sector, domain, discipline, pedagogy, school, society, civilization. The sequencing used in the scale is just a convenient way or organizing the strata and, by no mean, implies any kind of superiority of the high levels of the scale upon the low levels. The low to high sequencing merely captures the idea of taking into account institutions having broader and broader scopes.

According to Schneider (2013), the scale of levels is, among other things, what reveals ATD's scientific potential, because the crossing of different levels allows one to put to test the hypothesis drawn from ATD about a specific issue, antinomies in our case. Those hypotheses are thus falsifiable in the sense of Popper (1959). This turns the "specific issue", together with the hypothesis put forward within ATD's framework, into a didactical phenomenon and not just an opinion. We will use the word *phenomenotechnical* to qualify ATD's scientific potential as elaborated by Schneider (2013) who built on Bachelard's neologism phenomenotechnical found in his famous applied rationalism (1949).

INSTITUTIONAL CONSTRAINTS

Armed with the spirit of ATD, we now consider two institutions located at high levels of the scale (society and above). The first one is the modern mathematics reform and the second one the counter-reform to the modern one.

Institution I: the modern mathematics reform

Starting in the 30s, Bourbaki, a group of mathematicians, deeply changed mathematics. They helped to unify it relying on structures, spreading to the whole discipline, what van der Waerden (1930) had begun in algebra. The benefits were considerable and still have repercussions nowadays. They helped to pave the way to previously unsolved problems: Fermat's last theorem not to mention it. One key feature of structures is the ability to use a way of tackling a problem in a particular context and to transfer it to another that are governed by the same structure (Houzel, 2004). This initiative comes at a cost. Since then mathematics are based on more abstract concepts than ever before (Aczel, 2009). With Bourbaki, structures, the axiomatic method of Hilbert and deductive architectures have more than ever become part of what mathematics are.

[2] It is a complex theory with lots of concepts. We shall limit our presentation to those that fit our needs. The interested reader will find a detailed account of ATD in, for instance, Bosch & Gascon (2014).
[3] It should be stressed out that ATD's epistemology is to be considered as a *model* of knowledge built *to analyze didactical phenomena*. There is no claim that we should identify knowledge with institutional practices and that nothing else matters. In particular, there is no competition with platonism. We just feel the platonistic approach to didactical phenomena is unable to make sense of the antinomies we are dealing with.

This revolution led, together with other factors (economical, political, intellectual), to reform secondary school curricula, starting in the sixties (Houdebine, 1994). The modern mathematics reform was born. One strong trend found among the teaching principles was that abstract structures have to be taught from the start because they were considered to be "natural" for they are "isomorphic" to "structures" found in human minds (Charlot, 1984). Abstract group theory (Dienes, 1971) and set theory (Papy, 1963) are taught in secondary school and even in primary school in some cases.

The reform turned out to be a global disaster. Pupils could at best understand mathematics as a (meaningless) game based on (mathematical) words, they saw no connection to the "real" world (Houdebine, 1994).

Institution II: the counter-reform

By the end of the 70s, the modern mathematics reform was put to sleep. A counter-reform took place that intended to undo the damages of the reform. This broadly meant: make mathematics meaningful (again). The perceived collusion between Bourbaki's unifying effort and the reform led the counter-reform to somewhat go against what mathematics had turned into the 20th century. Set « theory » including notations were put aside as it was seen as one of the most meaningless aspect of the reform. Structures were also mostly put aside. The only remains were pieces of the theory of functions. Those pieces are the ones still present nowadays in secondary school we alluded to in the previous sections: injectivity, reciprocal, etc. The deductive architecture of mathematics, including proofs, became less important than the following principles.

- Mathematical concepts should be strongly linked to the « real » world
- Teaching should proceed from « concrete » to « abstract »
- Learning activities should be centered on pupils' background…

Putting it all together

We might be as much critical about the reform than the counter-reform. It is not our point here. Instead our goal is to explain how the articulation between these institutions allows us to make sense of the antinomies discussed in the previous sections.

In Belgium, to become a mathematics teacher, you first need to have a degree in mathematics[4] (5 years) and then you are trained to become a teacher during 1 year. So, a teacher foundation is first of all that of a non-professional mathematician: you learn about theories but you are not supposed to actually engage in any kind of research. The kind of mathematics they learned are mostly highly formalized following mathematics' evolution impacted by Bourbaki among others. So, to them, mathematics is all about structures, deductive architecture and proofs. This is what you mostly learn in courses: theorems and their proofs. Most teachers do not therefore consider the possibility of doing genuine (that is rigorous) mathematics another way.

But the counter-reform puts them in a paradox. How do they teach mathematics, "as they are", if their essence is frowned upon? They have to embrace nowadays students, the children of the counter-reform, and the somewhat contradictory constraints from institutions I and II. One way of dealing with such a situation is to use concepts that resemble their genuine mathematical counterparts. Those pedagogic concepts are "designed" for their ability to be given a "concrete" meaning.

For example, the reciprocal of a function's role, in the course, is not to transfer properties from exponentials to logarithmic functions. Such a use of reciprocals is deemed too "abstract" from the counter-reform point of view. The teachers thus have to resort to tricks to conciliate both institutions they are subordinated to. On the one hand, they have to make use of reciprocals because that is a sound mathematical way to proceed from the first institution's perspective. On the other hand, they have to ornament the reciprocals' mathematical use to fit it into the counter-reform ideology and the resulting school ecology. So, reciprocals are presented as something susceptible of a graphical interpretation (a symmetry between two graphs under $y = x$), because graphical interpretation are considered as a fundamental way of conveying meaning. Doing so, everything is fine in the classroom. Students understand what they see on the board because of its graphical nature and mathematics'

[4] In some cases, engineers and physicist are allowed to become mathematics teachers.

morality is safe because reciprocals were used: the reciprocal mathematical concept has been superseded with pedagogical one looking exactly the same at a distance.

Using this double play of concepts at the scale of an entire course results in a deductive architecture that is teared down when looking more carefully. This way of doing mathematics is not to be blamed as such on the teachers. Most of them do not see what is going on with such a course. They are, so to speak, forced into such doings unwillingly, due to the very nature of the ecology they are involved in. Teachers, in our high school that tried to stick to a strict formal mathematical course ended up into the sidings.

This way of doing mathematics "works" to a certain extent because it allows students to learn by heart and teachers to pretend they are giving a genuine mathematical course even if its level is very low. But it doesn't really work because concepts taken from the theory of functions such as reciprocals are not meant to be "concrete" the way they are concretized in the course. What do students really learn?

CONCLUSIONS

We have shown that the question of the high failure rates in the course, although of importance, is just the tip of the iceberg, that of the various institutions that impact what a teacher can or cannot do within a mathematical course. In our case, these institutions are the reform and the counter-reform. Taking into account these two allowed us to make sense of the antinomies found in the course and go beyond restrictive interpretations in terms of students' misbehavior and/or teachers' inabilities. The high levels of the scale of didactic codetermination convey a message. The impact of individuals, whether they are teachers or students, shouldn't be underestimated, but the institutional ecology to which they belong must be taken into account if we want to be able to understand various antinomies that cannot be accounted for relying solely on individual behaviors. In this sense, the didactic of mathematics has much insight to gain relying on anthropology and sociology. It is this junction that is put forward in ATD on which we have built on.

Although we feel these preliminary conclusions to be of interest, as mentioned in the introduction, our research isn't mature yet. We need to be able to position ourselves among the existing literature. It means the following questions should be addressed.

- Have similar conclusions been drawn in other countries?
- How do they articulate with ours if at all?
- If not, what does it mean?
- Does it invalidate our conclusions or does it put into light some peculiarities our high school's ecology?

Moreover we should look, as stated in our theoretical framework, at different levels of the scale and see if our hypothesis are consistent with other interpretations these other levels allow. This will be the subject of further investigations that will allow us to challenge the phenomenotechnicalty (Schneider, 2013) of our ongoing research.

RECOMMENDATIONS

Issuing recommendations at an early stage of a research is somewhat daring. Nevertheless, we feel this article has something to say about the relationship between teachers and culture. It is common in Belgium to find in initial teacher curricula several courses in epistemology, sociology, etc. that intend to broaden the future teachers' culture. This broadening is thought, among other things, to allow them to develop a reflective practice that is considered as an important part of today's best teaching practices (Beckers, 2009). We shall not discuss the relevance of reflective teaching but rather comment on the link between reflective teaching and culture. We feel that a background in epistemology, sociology, etc. does not guarantee that future teachers will engage in reflective teaching. One reason for this is that a course in say epistemology is more likely to constitute an initiation to different epistemologies. But between such a course and the daily routine of teachers there is a not so short distance. We can hardly expect teachers to fill in the gap. As this paper seems to demonstrate in a particular instance, flaws in their practice can result from high level constraints that are not easy to identify nor step aside because they form a context in which the whole education system is nested and they are moreover somewhat evanescent. We conclude from this, the relevance for a course in didactic, to train future teachers to use ATD and/or other didactic theories not as ideologies devised to "teach well" but as systemic and systematic tools designed to highlight the hypothesis underlying their teaching.

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