

Anyons in quantum mechanics with a minimal length

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Abstract

The existence of anyons, *i.e.* quantum states with an arbitrary spin, is a generic feature of standard quantum mechanics in $(2 + 1)$ -dimensional Minkowski spacetime. Here it is shown that relativistic anyons may exist also in quantum theories where a minimal length is present. The interplay between minimal length and arbitrary spin effects are discussed.

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I. INTRODUCTION

The existence of a minimal observable length in Nature is suggested by string theory and quantum gravity, see *e.g.* [1]. An economical way of introducing such a minimal length at the level of nonrelativistic quantum mechanics is to use a suitably modified Heisenberg algebra [2–4]. This proposition has been extended to covariant quantum mechanics in [5] and further investigated in many relevant cases including the Klein-Gordon [6] and Dirac [7] equations.

Although the study of quantum theories characterized by a minimal length has become an active field of research, some comments remain to be done about the case of a $(2 + 1)$ –dimensional spacetime with a minimal length. Indeed, it is well-known that spin is no longer quantized in $2 + 1$ dimensions, allowing the appearance of anyons, *i.e.* particles with arbitrary spin. To our knowledge, the question of the existence (or not) of anyons in a $(2 + 1)$ –dimensional spacetime with a minimal length has never been addressed. It is the topic of the present paper.

II. POINCARÉ ALGEBRA IN $2 + 1$ DIMENSIONS

A. Generalities

The Poincaré algebra in $2 + 1$ dimensions can be written under the form [8]

$$[J^\mu, J^\nu] = -i\varepsilon^{\mu\nu\rho} J_\rho, \quad [J^\mu, P^\nu] = -i\varepsilon^{\mu\nu\rho} P_\rho, \quad [P^\mu, P^\nu] = 0, \quad (1)$$

where P^μ are the translation generators and where

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho} L_{\nu\rho}, \quad (2)$$

$L_{\mu\nu}$ being the Lorentz generators. The convention $\epsilon^{012} = 1$ is used, as well as the Minkowski metric $\eta = \text{diag}(+, -, -)$. The two Casimir operators associated to (1) read, for massive representations,

$$M^2 = \mathbf{P}^2, \quad s = \frac{\mathbf{P} \cdot \mathbf{J}}{M}. \quad (3)$$

They represent the squared mass and the spin of a given state respectively, while the notation $\mathbf{X} = (X^0, X^1, X^2)$ for a 3–vector.

It is useful to note that, in the rest frame of a given state, s reduces to $J^0 = \pm L_{12}$, that is the generator of spatial rotations. One recovers then a maybe more intuitive definition of spin as the phase factor associated to rotations.

The unitary irreducible representations of the Lorentz algebra in $2 + 1$ dimensions – the first commutator of (1) – have been built in [8], while the unitary irreducible representations of the full Poincaré algebra have been obtained in [9]. The conclusions shared by both studies are identical in what concerns the spin of the representations: It can be an arbitrary real number. This is a peculiar feature of quantum mechanics in $2 + 1$ dimensions and strongly differs from the $(3 + 1)$ – dimensional case, in which the spin of a state can only be integer or half-integer. The properties of such particles with arbitrary spin, called anyons, have been studied in a considerable amount of works. The interested reader may find relevant informations in the pioneering works [10] and [11, 12], as well as in the reviews [13, 14].

It is known that finite-dimensional representations of the Lorentz algebra are non-unitary in $2 + 1$ dimensions. However, this does not imply that bosons and fermions are forbidden. It is indeed worth mentioning that bosonic and fermionic states can be built in $2 + 1$ dimensions from the tensor product of non-unitary infinite-dimensional representations of the Lorentz algebra. That subtle issue has been studied in [15], to which we refer the interested reader.

B. Minimal length representation of the Poincaré algebra

It has been shown in [5] that, for a $(D + 1)$ –dimensional spacetime with metric $\eta = \text{diag}(+, -, \dots, -)$, the Poincaré algebra is represented by the generators

$$P_\mu = (1 - \beta \hat{\mathbf{P}}^2)^{-1} \hat{P}_\mu, \quad L_{\mu\nu} = (1 - \beta \hat{\mathbf{P}}^2)^{-1} (\hat{P}_\nu \hat{X}_\mu - \hat{P}_\mu \hat{X}_\nu) \quad (4)$$

provided that

$$\begin{aligned} [\hat{X}^\mu, \hat{P}^\nu] &= -i \left[(1 - \beta \hat{\mathbf{P}}^2) \eta^{\mu\nu} - \beta' \hat{P}^\mu \hat{P}^\nu \right], \\ [\hat{X}^\mu, \hat{X}^\nu] &= -i \left[(2\beta - \beta') - (2\beta + \beta') \beta \hat{\mathbf{P}}^2 \right] L^{\mu\nu}, \\ [\hat{P}^\mu, \hat{P}^\nu] &= 0. \end{aligned} \quad (5)$$

where it is assumed that $\beta, \beta' \in \mathbb{R}^+$. The modified Heisenberg algebra (5) leads to a minimal uncertainty on the measurement of a position that can be computed to be, assuming isotropicity [5],

$$\Delta X_{\min} = \sqrt{(D\beta + \beta')(1 - \beta \langle (P^0)^2 \rangle)}, \quad (6)$$

and is usually referred to as “minimal length”.

III. RELATIVISTIC ANYONS WITH A MINIMAL LENGTH

It follows from the general results recalled in the previous section that, in $(2 + 1)$ dimensions, quantum theories which are both Poincaré invariant and formulated in a spacetime with minimal length are allowed. As a consequence, anyons may exist when a minimal length is present too. Wave equations describing an anyon $|\psi\rangle$ have been proposed in [16] in a form that is convenient regarding to our framework. They read

$$V_\mu|\psi\rangle = 0, \quad V_\mu = sP_\mu - i\epsilon_{\mu\nu\rho}P^\nu J^\rho + MJ_\mu. \quad (7)$$

They are valid whatever the representation chosen for P^μ and J^μ , so they can be used with the minimal length representation (4), (5). An extensive discussions of the solutions of (7) can be found in [16].

In view of explicit computations, it is convenient to note that the algebra (5) can be represented by the operators [5]

$$\begin{aligned} \hat{P}^\mu &= p^\mu, \\ \hat{X}^\mu &= (1 - \beta\mathbf{p}^2)x^\mu - \beta'p^\mu\mathbf{p} \cdot \mathbf{x} + i\gamma p^\mu, \end{aligned} \quad (8)$$

where $x_\mu = -i\frac{\partial}{\partial p^\mu}$ and \mathbf{p} are standard position and momentum operators, *i.e.* $[x^\mu, p^\nu] = -i\eta^{\mu\nu}$. A straightforward computation shows that, using this representation, the Lorentz generators take a more familiar form

$$L_{\mu\nu} = i(p_\mu\frac{\partial}{\partial p^\nu} - p_\nu\frac{\partial}{\partial p^\mu}). \quad (9)$$

Using the representation (8) in polar coordinates, namely $p^\mu = (p^0, p, \theta_p)$, one quickly shows that

$$J^0 = L_{12} = i\frac{\partial}{\partial\theta_p}, \quad (10)$$

recovering a standard form for J^0 even if a modified Heisenberg algebra is considered.

A peculiarity of algebra (5) is that $[\hat{X}^\mu, \hat{X}^\nu] \propto L^{\mu\nu}$. Hence, the noncommutativity of spatial coordinates is such that $[\hat{X}^1, \hat{X}^2] \propto J^0$. Let us indeed consider an anyonic state of

squared mass M^2 and spin s in its rest frame and denote $|\psi; M^2, s, j\rangle$ such a state. Then one is led to the following uncertainty relation on the spatial coordinates:

$$\Delta X^1 \Delta X^2 \geq \frac{1}{2} |(2\beta - \beta') - (2\beta + \beta')\beta M^2| s. \quad (11)$$

Anyons with larger spins then “feel” a larger spatial noncommutativity. Note however that, even for small values of s , one has the lower bound $\Delta X^1 \Delta X^2 \geq \Delta X_{\min}^2$ implied by the modified Heisenberg algebra.

IV. NONRELATIVISTIC LIMIT

The modified Heisenberg algebra initially proposed by Kempf in a nonrelativistic version [2] can be recovered from algebra (5) and from (4) by neglecting the terms in $\beta(P^0)^2$, *i.e.* by assuming that the typical energy of the system under study is negligible compared to $1/\sqrt{\beta}$, and by only considering the spatial commutators. The only nonvanishing commutators in the case of two spatial dimensions are then

$$\begin{aligned} [\hat{X}^i, \hat{P}^j] &= i \left[(1 + \beta \hat{P}^2) \delta^{ij} + \beta' \hat{P}^i \hat{P}^j \right], \\ [\hat{X}^1, \hat{X}^2] &= -i \left[(2\beta - \beta') + (2\beta + \beta') \beta \hat{P}^2 \right] J^0, \end{aligned} \quad (12)$$

where $\hat{P}^2 = (P^1)^2 + (P^2)^2$. The momentum representation (8) has the nonrelativistic counterpart

$$\begin{aligned} \hat{P}^j &= p^j, \\ \hat{X}^j &= i(1 + \beta p^2) \frac{\partial}{\partial p^j} + i\beta' p^j \vec{p} \cdot \vec{\nabla}_p + i\gamma p^j. \end{aligned} \quad (13)$$

It can be checked from the above representation that any Hamiltonian of the form $T(\hat{P}^2) + V(\hat{X}^2)$, being spherically symmetric in momentum representation, will commute with J^0 , according to (10). Moreover, in the rest frame of the eigenstate $|\chi\rangle$ under study, J^0 is identified with the spin; its eigenvalue s can then be arbitrary. It follows

$$\langle \vec{p} | \chi \rangle = e^{is\theta_p} \psi(p), \quad (14)$$

which fixes the angular dependence of the state. This last relation also shows that, if \vec{P} and \vec{X} are the relative momentum and position of a two-body system made of two identical bodies, the wave function acquires a phase factor $e^{is\pi}$ under permutation of the two bodies. This

exotic phase factor is the signature of a braid statistics, in agreement with the generalized spin-statistics theorem [17].

As an illustration, let us consider the Hamiltonian $\frac{\hat{P}^2}{2\mu} + \frac{1}{2}\mu\omega^2\hat{X}^2$, whose energy spectrum is analytically known in D spatial dimensions [18]. When $D = 2$, the spin is arbitrary and, using the results of this last reference, the energy spectrum is given by

$$\begin{aligned} \frac{E}{\omega} = & (2n + |s| + 1) \sqrt{1 + \left[\beta^2 s^2 + \frac{(2\beta + \beta')^2}{4} \right] \mu^2 \omega^2} \\ & + [(\beta + \beta')(2n + |s| + 1)^2 + (\beta - \beta')(s^2 + 1) + \beta'] \frac{\mu\omega}{2}, \end{aligned} \quad (15)$$

which is a generalization of the anyonic harmonic oscillator, that can be recovered with $\beta = \beta' = 0$, see *e.g.* [14, 19]. The main effect of the modified Heisenberg algebra is to break to well-known degeneracy in $2n + |s| + 1$.

It is worth pointing out that there exists a deformation of the Heisenberg algebra which is inequivalent to (12) but which can also be related to anyons. It reads

$$\begin{aligned} [a^-, a^+] &= 1 + \nu K, \\ \{K, a^\pm\} &= 0, \quad K^2 = 1, \end{aligned}$$

where the real parameter ν and the kleinian K are responsible for the deformation. a^\pm are creation and annihilation operators. As shown in [20], this algebra can be used to describe in an elegant way relativistic quantum states with arbitrary spin. In the nonrelativistic limit, anyons built from such a formalism reduce to free massive particles in the noncommutative plane [21]: As in Eq. (11), states with arbitrary spin are indeed associated to noncommutative spatial coordinates.

V. SUMMARY AND OUTLOOK

Anyons may exist in $2 + 1$ dimensions as representations of the Poincaré algebra. It follows that they may be present in any Poincaré invariant spacetime, including those where a minimal length is present. In such spacetimes, an anyon wave equation can be defined.

As an outlook, let us mention that previously known analytical results about the spectra of Schrödinger-like Hamiltonians with modified Heisenberg algebra could be re-analysed in the particular case of two spatial dimensions to better appraise the interplay between of an arbitrary spin and a minimal length.

An other way of producing anyons is to minimally couple a particle to a vortex-like gauge field [11]. It could be interesting to see what is the generalization of such a gauge field configuration in electrodynamic with a minimal length, which has been proposed recently in [22]; we leave such a task for future works.

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